THE KINEMATICS OF VISCOELASTIC MEDIA WITH FINITE DEFORMATIONS

PMM Vol. 32, No. 2, 1968, pp. 217-231

Iu. E. BUEVICH
(Moscow)

(Received February 8, 1967)

The kinematics of viscoelastic fluids with additive deformations is considered. A kinematic relationship is obtained between the total and viscous strain rate tensors and the elastic strain tensor, which underlies the axiomatic construction of rheological models of such fluids. A representation is found for the components of the elastic strain rate tensor in terms of observed quantities, which permits an analysis of the change in different scalar characteristics due to a change in the elastic state of the fluid. The developed method is also applicable to the investigation of other continua with complex deformations. The obtained results are later applied to construction of a nonlinear rheological model of a compressible viscoelastic fluid of Maxwell type.

The viscoelastic media considered below, whose total deformation at an arbitrary time can be represented as the sum of elastic (reversible) and viscous (irreversible) deformations. The viscous deformations are hence defined as residual deformations with instantaneous liberation of the medium from all elastic deformations. The question of the physical realizability of such a process is discussed below, and it is meanwhile sufficient that it can always be done meaningfully in a small neighborhood of the considered material element. Fixing some initial state in which the viscous deformations are assumed zero, the elastic components of the total deformation of the medium can be determined by an analogous means.

1. Following Sedov [1], let us define two coordinate systems in the volume filled with viscoelastic fluid: a fixed Eulerian reference system $\{x^i\}$ with basis vectors $\ni_i$ and a Lagrange convective coordinate system $\{\xi^i\}$ frozen in the continuum element in such a manner that the values of ξ^i corresponding to different individual points of the medium, are time-independent. Investigating only the total deformations of the medium, we consider two positions of the continuum relative to the $\{x^i\}$ coordinate system, to which two different bases of the Lagrange coordinate system $\{\xi^i\}$ correspond:

1. An initial position with basis vectors $\ni_i^{(1)}$; in this basis definite fixed points correspond to individual points of the medium.

2. A position of the deformable medium with basis vectors $\ni_i^{(2)}$. The disposition of the vectors $\ni_i^{(2)}$ relative to $\ni_i^{(1)}$ at a certain time determines the total deformation of the medium accumulated up to this time.

Motions of continuum elements associated only with the viscous or only with the elastic components of the total deformations may also be considered formally. This permits the analysis of still two other 'intermediate' positions of the continuum elements, to which two other bases of the Lagrange convective coordinate system will correspond [1].

1. A position of the medium which would experience only, viscous deformations, with basis vectors $\ni_i^{(3)}$. The basis $\ni_i^{(3)}$ plays the part of initial basis with respect to elastic

deformations and the part of final basis with respect to the viscous deformations. The disposition of the vectors $\ni_i^{(3)}$ relative to $\ni_i^{(1)}$ determines the viscous deformations of the medium, and the disposition of $\ni_i^{(2)}$ relative to $\ni_i^{(3)}$ the elastic deformations. At each instant the position $^{(3)}$ of the medium is obtained from the position $^{(2)}$ by the process of instantaneous liberation of the medium from all elastic deformations.

2. A position $^{(4)}$ of the medium, which undergoes the elastic total deformation components, can also be considered analogously to the position $^{(3)}$. The corresponding basis vectors $\ni_i^{(4)}$ are the initial basis for the viscous, and the final basis for the elastic deformations.

It is clear that representations of individual points of the medium with different meanings, correspond to the positions $^{(2)}$, $^{(3)}$, $^{(4)}$ of the medium.

For the lengths of a small segment composed of the same continuum points, we have at different locations

$$d\mathbf{r}_k = d\xi^i \ni_i^{(k)}, \quad ds_k^2 = g_{ij}^{(k)} d\xi^i d\xi^j, \quad g_{ij}^{(k)} = (\ni_i^{(k)} \ni_j^{(k)}), \quad k = 1, 2, 3, 4$$

In conformity with the presence of four foundamental forms, the various tensors may be considered in four spaces governed by the various positions of the continuum elements and corresponding to the different bases of the Lagrange coordinate system which have been introduced[1]. Let us establish the correspondence

$$ds_2^2 = g_{ij}^{(2)} d\xi^i d\xi^j = g_{ij} dx^i dx^j, \quad ds_3^2 = g_{ij}^{(3)} d\xi^i d\xi^j = g_{ij} dx^{[i]} dx^{[j]}$$
$$ds_4^2 = g_{ij}^{(4)} d\xi^i d\xi^j = g_{ij} dx^{(i)} dx^{(j)}, \qquad g_{ij} = (\ni_i \ni_j)$$

Hence, it is seen that all the tensors introduced in spaces $^{(2)}$, $^{(3)}$ and $^{(4)}$ can be considered simultaneously as tensors defined in one basis $\ni_i$, but as before, in different spaces corresponding to the fundamental forms ds_2^2, ds_3^2 and ds_4^2. The quantities $dx^{(i)}$ and $dx^{[i]}$ play the part of components of the vectors $d\mathbf{r}_4$ and $d\mathbf{r}_3$ in the basis $\ni_i$. Similarly, all the tensors may also be referred to any other basis.

Using the bases $\ni_i^{(1)}$, $\ni_i^{(2)}$ and $\ni_i^{(3)}$, let us introduce the total E, the elastic (E), and the viscous [E] strain tensors by utilizing the relationships

$$\mathbf{E} = \varepsilon_{ij} \ni_2^i \ni_2^j = e_{ij} \ni^i \ni^j, \qquad \mathbf{E}_0 = \varepsilon_{ij} \ni_1^i \ni_1^j$$
$$(\mathbf{E}) = \varepsilon_{(ij)} \ni_2^i \ni_2^j = e_{(ij)} \ni^i \ni^j, \qquad (\mathbf{E})_0 = \varepsilon_{(ij)} \ni_3^i \ni_3^j$$
$$[\mathbf{E}] = \varepsilon_{[ij]} \ni_3^i \ni_3^j = e_{[ij]} \ni^i \ni^j, \qquad [\mathbf{E}]_0 = \varepsilon_{[ij]} \ni_1^i \ni_1^j \tag{1.1}$$
$$\varepsilon_{ij} = {}^1/_2 (g_{ij}^{(2)} - g_{ij}^{(1)}), \quad \varepsilon_{(ij)} = {}^1/_2 (g_{ij}^{(2)} - g_{ij}^{(3)}), \quad \varepsilon_{[ij]} = {}^1/_2 (g_{ij}^{(3)} - g_{ij}^{(1)})$$

Here the subscript zero refers to the initial-state spaces for the appropriate deformations. The quantities e_{ij}, $e_{(ij)}$, $e_{[ij]}$ and ε_{ij}, $\varepsilon_{(ij)}$, $\varepsilon_{[ij]}$ can be considered as functions of t and ξ^m or t and x^m. The Jacobian of the appropriate transformation is assumed not to be zero. Other tensors associated with the total, elastic and viscous strains in different bases and spaces may be determined analogously.

Using the second possible method of describing the medium deformation, and introducing the bases $\ni_i^{(1)}$, $\ni_i^{(2)}$, $\ni_i^{(4)}$, we obtain instead of (1.1)

$$(\mathbf{E}') = \varepsilon'_{(ij)} \ni_4^i \ni_4^j = e_{(ij)} \ni^i \ni^j, \qquad (\mathbf{E}')_0 = \varepsilon'_{(ij)} \ni_1^i \ni_1^j$$
$$[\mathbf{E}'] = \varepsilon'_{[ij]} \ni_2^i \ni_2^j = e_{[ij]} \ni^i \ni^j, \qquad [\mathbf{E}']_0 = \varepsilon'_{[ij]} \ni_4^i \ni_4^j \tag{1.2}$$
$$\varepsilon'_{(ij)} = {}^1/_2 (g_{ij}^{(4)} - g_{ij}^{(1)}), \qquad \varepsilon'_{[ij]} = {}^1/_2 (g_{ij}^{(2)} - g_{ij}^{(4)})$$

The previous expressions in (1.1) are valid for $\mathbf{E}$, $\mathbf{E}_0$ and $\varepsilon_{[ij]}$.

Let us note that, in general, $\varepsilon_{(ij)} \neq \varepsilon_{(ij)}'$ and $\varepsilon_{[ij]} \neq \varepsilon_{[ij]}'$. These assertions follow directly from the noncommutativity property of finite deformations, and they correspond physically to the fact that the tensors (E) and (E′) characterize the state of different material fibers. The components of these tensors in the Euler basis should be the same because of the physical uniqueness of the deformation process. For definiteness, only the first method of describing the deformation is used below as being more natural for the formulation of rheological models of viscoelastic fluids.

2. According to the definition (1.1), an identity expressing mathematically the postulate of additivity of the deformations

$$\varepsilon_{ij} = \varepsilon_{(ij)} + \varepsilon_{[ij]} \tag{2.1}$$

holds for the tensor components ε_{ij}, $\varepsilon_{(ij)}$ and $\varepsilon_{[ij]}$.

This relationship has no tensor character in the sense that it is not satisfied for contravariant or mixed components of the strain tensors. This is associated with the fact that different metric tensors must be used to raise the indices in the different terms of (2.1). However, upon passing from $\{\xi^i\}$ to any other Lagrange coordinate system, (2.1) transforms according to the customary tensor rules. Let us note that the additivity relationship (2.1) does not generally hold for the quantities e_{ij}, $e_{(ij)}$ and $e_{[ij]}$.

Considering (2.1) at close times, we obtain an equation for the increments of the components ε_{ij}, $\varepsilon_{(ij)}$ and $\varepsilon_{[ij]}$ in the time dt:

$$d\varepsilon_{ij} = d\varepsilon_{(ij)} + d\varepsilon_{[ij]}$$

$$d\varepsilon_{ij} = \frac{D\varepsilon_{ij}}{Dt}dt, \quad d\varepsilon_{(ij)} = \frac{D\varepsilon_{(ij)}}{Dt}dt, \quad d\varepsilon_{[ij]} = \frac{D\varepsilon_{[ij]}}{Dt}dt \tag{2.2}$$

Here $D(\ldots)/Dt$ is the symbol of convective differentiation of the tensor component with respect to time (for constant ξ^m). Definitions of $d\varepsilon_{ij}$, $d\varepsilon_{(ij)}$ and $d\varepsilon_{[ij]}$ in terms of the corresponding derivatives in (2.2) follow from the results in [1]. Let us note that these derivatives are taken relative to different positions of the deformable continuum, which is manifested in the definitions of the corresponding tensor components in the Euler basis.

Thus, by utilizing known rules of transforming convective tensor derivatives with respect to the time [1 and 2], we obtain for the total strain rate tensor and for the quantities $d\varepsilon_{ij}$ the following Formula:

$$\Gamma = \gamma_{ij}^{(2)}\partial_2{}^i\partial_2{}^j = (D(\ldots)/Dt)(\varepsilon_{ij}\partial_2{}^i\partial_2{}^j) - \gamma_{ij}\partial^i\partial^j = \{(\partial(\ldots)/dt + v^m\nabla_m)e_{ij} + (\nabla_i v^m)e_{mj} + e_{im}\nabla_j v^m\}\partial^i\partial^j, \qquad d\varepsilon_{ij} = \gamma_{ij}^{(2)}dt$$

An analogous formula holds for the tensor $(\Gamma)^*$ which has the components $D\varepsilon_{(ij)}/Dt$ in the basis $\partial_i{}^{(2)}$, and for the increments $d\varepsilon_{(ij)}$ expressed in terms of these components.

Converely, for the viscous strain rate tensor and the quantities $d\varepsilon_{[ij]}$ we obtain the formal representations

$$[\Gamma] = \gamma_{[ij]}^{(3)}\partial_3{}^i\partial_3{}^j = (D(\cdots)/Dt)\,\varepsilon_{[ij]}\partial_3{}^i\partial_3{}^j = \gamma_{[ij]}\partial^i\partial^j =$$

$$= \left\{(D(\cdots)/Dt)\,e_{[ij]} + \frac{\partial v^{[m]}}{\partial\xi^i}e_{[mj]} + e_{[im]}\frac{\partial v^{[m]}}{\partial\xi^j}\right\}\partial^i\partial^j, \qquad de_{[ij]} = \gamma_{[ij]}^{(3)}dt$$

Here $v^{[m]}$ are velocity components associated with the viscous fluid displacements. As is known [1], the partition of the total velocity v^i, as well as the antisymmetric tensor ω_{ij} corresponding to the total vortex vector, into elastic $v^{(i)}$, $\omega_{(ij)}$ and viscous $v^{[i]}$, $\omega_{[ij]}$ components is not unique. However, the quantities $\gamma_{(ij)}$ and $\gamma_{[ij]}$ are uniquely defined. Only under this condition is it possible to speak of a specific physical (elastic, say) state of a medium.

Therefore, from (2.2) we obtain the fundamental kinematic relationship for the strain rates

$$(D\varepsilon_{ij}/Dt)\,\varepsilon_{(ij)} + \gamma_{[ij]}^{(3)} = \gamma_{ij}^{(2)} \tag{2.3}$$

Let us emphasize that the $\gamma_{[ij]}^{(3)}$ in this relationship are viscous strain rate tensor components in the basis $\partial_i^{(3)}$, and all the other quantities in (2.3) are tensor components in the basis $\partial_i^{(2)}$.

Let us introduce $d\eta^i$ as components of the vector $d\mathbf{r}_3 = d\xi^i \partial_i^{(3)}$ in the basis $\partial_i^{(2)}$. The quantities $d\eta^i$ play the same part, in principle, as do the $dx^{(i)}$, and $dx^{[i]}$, introduced above. Introduction of $d\eta^i$ permits transformation of tensors defined in a space corresponding to position $^{(3)}$ of the deformable continuum, to the basis $\partial_i^{(2)}$ exactly as the introduction of $dx^{(i)}$ and $dx^{[i]}$ permitted transformation of the tensors defined in spaces $^{(3)}$ or $^{(4)}$ to the Euler basis. The quantities $d\eta^i$ and $d\xi^i$ are connected by affine transformation formulas [1]

$$d\eta^i = C^{i\cdot}_{\cdot m}\, d\xi^m$$

The matrix $\mathbf{C}$ entering here may be considered as a matrix defining the transformation of the basis vectors $\partial_i^{(3)} \to \partial_i^{(2)}$, referred to the same space. For it we have the representation [1]

$$\mathbf{C} = e^{\mathbf{K}}\,(\mathbf{g}^{(2)} - 2\,\boldsymbol{\varepsilon})^{1/2}, \quad \mathbf{C}' = (\mathbf{g}^{(2)} - 2\,\boldsymbol{\varepsilon})^{1/2} e^{-\mathbf{K}}, \quad \mathbf{g}^{(2)} = \| g_{ij}^{(2)} \|, \quad \boldsymbol{\varepsilon} = \| \varepsilon_{(ij)} \| \qquad (2.4)$$

Here $\mathbf{K}$ is an antisymmetric matrix corresponding to the vector of rotation of the principal elastic deformation axes, and the prime denotes the transposition operation.

We have the following representation for the viscous strain rate tensor components $\gamma_{[ij]}^{(3)}$ in the basis $\partial_i^{(3)}$ in terms of components of this tensor in the basis $\partial_i^{(2)}$:

$$\boldsymbol{\gamma}^{(3)} = \mathbf{C}'\boldsymbol{\gamma}^{(2)}\mathbf{C}, \qquad \boldsymbol{\gamma}^{(k)} = \| \gamma_{[ij]}^{(k)} \| \qquad (k = 2, 3)$$

Substituting this into (2.3) we obtain a relationship which may be considered as a tensor

$$(D(\cdots)/Dt)\,\varepsilon_{(ij)} + C'^{\cdot m}_{i\cdot}\,\gamma_{[mn]}^{(2)}\,C^{n\cdot}_{\cdot j} = \gamma_{ij}^{(2)} \qquad (2.5)$$

The kinematic relationship (2.5) plays a fundamental part in the formulation of invariant rheological equations.

3. In addition to (2.5), the relationship [1]

$$\gamma_{(ij)} + \gamma_{[ij]} = \gamma_{ij}, \qquad \gamma_{ij} = {}^1/_2\,(\nabla_i v_j + \nabla_j v_i)$$
$$\gamma_{(ij)} = {}^1/_2\,(\nabla_i v_{(j)} + \nabla_j v_{(i)}), \quad \gamma_{[ij]} = {}^1/_2\,(\nabla_i v_{[j]} + \nabla_j v_{[i]}) \qquad (3.1)$$

can be obtained independently.

The equality (3.1) may also be considered as a tensor relationship, and in particular, it can be written for the strain rate tensor components in the basis $\partial_i^{(2)}$. Evidently, relationships (3.1) and (2.5) should agree because they contain the same information on the motion of the medium. From (2.5) and (3.1) we then obtain the following tensor Eq.

$$\gamma_{(ij)}^{(2)} = (D(\cdots)/Dt)\,\varepsilon_{(ij)} + C'^{\cdot m}_{i\cdot}\gamma_{[mn]}^{(2)}C^{n\cdot}_{\cdot j} - \gamma_{[ij]}^{(2)} \qquad (3.2)$$

The quantities $\gamma_{(ij)}^{(2)}$ are components of the elastic strain rate tensor (Γ) in the basis $\partial_i^{(2)}$. As is easily seen, the tensor (Γ) differs from the tensor $(\Gamma)^*$ introduced earlier, which is the kinematic characteristic of the motion associated with a change in both the elastic and viscous strains. This is because the tensor (E) itself describes the true elastic state of the material elements of the medium ambiguously.

In fact, let us introduce the length elements $ds_k{}^i$ $(k = 2{,}3)$ along the i-th principal axis of the tensors $(\mathrm{E})_*$ and (E) in appropriate spaces. It is clear that the differences $ds_2{}^i - ds_3{}^i$ are uniquely defined by the components $\varepsilon_{(ii)}$ of the tensors $(\mathrm{E})_*$ and (E) in these axes. The elastic state of the medium is defined uniquely by values of the principal elongations [1]

$$\overset{*}{\varkappa}_{(i)} = \frac{ds_2{}^i - ds_3{}^i}{ds_3{}^i}, \qquad \varkappa_{(i)} = \frac{ds_2{}^i - ds_3{}^i}{ds_3{}^i}$$

or the eigenvalues

$$\varepsilon_{(i)}^{\circ} = g_3^{ii}\varepsilon_{(ii)}, \qquad \varepsilon_{(i)} = g_2^{ii}\varepsilon_{(ii)}, \qquad (i = 1, 2, 3)$$

of the tensors $(\mathbf{E})_0$ and $(\mathbf{E})$, on which the quantities $\varkappa_{(i)}^{\circ}$ and $\varkappa_{(i)}$ depend uniquely. It is easy to see that these quantities are functions not only of the differences $ds_2^i - ds_3^i$, but also of the elements $ds_3^i - ds_2^i$, which are determined equally by the development of viscous strains in the system. Hence, the quantities $\varepsilon_{(ij)}$ may be considered as representative characteristics of the elastic state of the medium only when the viscous motions do not result in changes in the elementary lengths in the material, i.e., $\gamma_{[ij]} \equiv 0$. Correct expressions are obtained for $\gamma_{(ij)}$, evidently, if $\varepsilon_{(ij)}$ is differentiated with the condition of invariant lengths of the vectors $\partial_i^{(3)}$, as well as the condition of constant ξ^m, is imposed, or equivalently, the condition of constant diagonal components of the metric tensor g_3^{ii} in the principal axes of the tensors $(\mathbf{E})_0$, $(\mathbf{E})$. Let us note that rotations of the vectors $\partial_i^{(3)}$ relative to $\partial_i^{(1)}$ are not fixed in such a differentiation, and may be arbitrary because they do not affect the change in material lengths.

Using reasoning similar to that which was used in introducing the Jaumann derivative, we see that the correct values of $\lambda_{(ij)}$ and $\gamma_{(ij)}^{(2)}$ are obtained if it is formally considered that $\gamma_{ij} = \gamma_{(ij)}$ in differentiating $\varepsilon_{(ij)}$ and later passing to the Euler basis. We hence have the following equation for $\gamma_{(ij)}$ in the Euler basis:

$$\begin{gathered}\gamma_{(ij)} = (\partial/\partial t + v^m \nabla_m) e_{(ij)} + e_{(im)}\gamma_{(\cdot j)}^{(m\cdot)} + \gamma_{(i\cdot)}^{(\cdot m)} e_{(mj)} + \\ + e_{(im)}\omega_{\cdot j}^{m\cdot} + \omega_{i\cdot}^{\cdot m} e_{(mj)}, \qquad \omega_{ij} = {}^1/_2(\nabla_i v_j - \nabla_j v_i)\end{gathered} \tag{3.3}$$

The following representation

$$\gamma_{(ij)}^{(2)} = (D(\cdot\cdot\cdot)/Dt)\,\varepsilon_{(ij)} - \varepsilon_{(im)}\gamma_{[\cdot j]}^{(2)[\cdot m]} - \gamma_{[i\cdot]}^{(2)[m\cdot]}\varepsilon_{(mj)} \tag{3.4}$$

corresponds to (3.3) in the Lagrange basis $\partial_i^{(2)}$.

Let us compare (3.2) and (3.4). To do this, we represent the matrices $\mathbf{C}$ and $\mathbf{C}'$ from (2.4) as

$$\begin{gathered}\mathbf{C} = (\mathbf{g}^{(2)} - 2\,\boldsymbol{\varepsilon})^{1/2} + (e^{\mathbf{K}} - \mathbf{g}^{(2)})(\mathbf{g}^{(2)} - 2\,\boldsymbol{\varepsilon})^{1/2} \\ \mathbf{C}' = (\mathbf{g}^{(2)} - 2\,\boldsymbol{\varepsilon})^{1/2} + (\mathbf{g}^{(2)} - 2\,\boldsymbol{\varepsilon})^{1/2}(e^{-\mathbf{K}} - \mathbf{g}^{(2)})\end{gathered}$$

The first members in these expressions are independent of $\mathbf{K}$, the second vanish as $\mathbf{K} \to 0$. We have the matrix equality

$$\begin{gathered}\mathbf{C}'\boldsymbol{\gamma}^{(2)}\mathbf{C} = (\mathbf{g}^{(2)} - 2\,\boldsymbol{\varepsilon})^{1/2}[\boldsymbol{\gamma}^{(2)} + \boldsymbol{\gamma}^{(2)}(e^{\mathbf{K}} - \mathbf{g}^{(2)}) + (e^{-\mathbf{K}} - \mathbf{g}^{(2)})\boldsymbol{\gamma}^{(2)} + \\ + (e^{-\mathbf{K}} - \mathbf{g}^{(2)})\boldsymbol{\gamma}^{(2)}(e^{\mathbf{K}} - \mathbf{g}^{(2)})](\mathbf{g}^{(2)} - 2\,\boldsymbol{\varepsilon})^{1/2}\end{gathered}$$

In constructing rheological models the tensors $(\mathbf{E})$ and $[\Gamma]$ are expressed as tensor functions of the same tensor $\mathbf{T}$ by using some kind of postulates. Hence, the matrices $\boldsymbol{\varepsilon}$ and $\gamma^{(2)}$ may be considered commutative. Then a comparison between (3.2) and (3.4) results in the Eq.

$$\begin{gathered}(\mathbf{g}^{(2)} - 2\,\boldsymbol{\varepsilon})^{1/2}[\boldsymbol{\gamma}^{(2)}(e^{\mathbf{K}} - \mathbf{g}^{(2)}) + (e^{-\mathbf{K}} - \mathbf{g}^{(2)})\boldsymbol{\gamma}^{(2)} + \\ + (e^{-\mathbf{K}} - \mathbf{g}^{(2)})\boldsymbol{\gamma}^{(2)}(e^{\mathbf{K}} - \mathbf{g}^{(2)})](\mathbf{g}^{(2)} - 2\,\boldsymbol{\varepsilon})^{1/2} = 0\end{gathered} \tag{3.5}$$

Eq. (3.5) imposes definite constraints on the possible deformations in the system. Roughly speaking, (3.5) plays the part of the known compatibility equation for elastic (or viscous) deformation taken separately. In the general case of finite deformations (3.5) differs from that customarily used in the geometrically nonlinear theory. Hence, the bases $\partial_i^{(3)}$ and $\partial_i^{(4)}$ are generally introduced into noneuclidean curved spaces corresponding to the situations of a deformable $^{(3)}$ and $^{(4)}$. Violation of the euclidean conditions is equivalent to violation of the continuity of the medium under the above-mentioned conceptual process of freeing the medium from elastic strains.

It is easy to see that the unloading process for which the viscous strains remain unchanged, is not realizable without violating the continuity of the medium. In fact, let us consider the complex of state of a viscoelastic fluid with additive strains. The state of a system consisting of a viscous fluid and an elastic strained sample submerged therein is a rough analog of such a state.

Upon removal of the elastic strains, and under the condition that the position of the individual viscous fluid elements does not change here, we arrive at a state in which there are cavities free of material, as well as sections which simultaneously contain both, fluid and elastic body. In other words, we arrive at a space having 'holes', which indeed make it noneuclidean. Analogous reasoning on the euclidean conditions of an 'intermediate' space is expounded in the modern theory of plasticity [1].

In proposing conservation of the continuity of the medium during the actual unloading process as a necessary condition, we see that the removal of stresses is possible only after a finite nonzero time interval since this process is inevitably accompanied by additional viscous strains. From the physical viewpoint, this means that only instantaneous removal of the external forces acting on the system is actually realizable. If a basis $\ni_i'$, describing the position of the continuum after the actual unloading process, is introduced, then the unloading process itself may be interpreted as the motion of the bases $\ni_i^{(3)}$ and $\ni_i^{(2)}$ towards the basis $\ni_i'$ until coincidence. It follows from the above that the conseptual process used above for the liberation of the medium from elastic strains can generally be consistently determined only locally.

Let us note that (3.5), and therefore, the representations (3.3) and (3.4) for the elastic strain rate tensor components also, can be obtained by completely independent means from (2.5) and (3.2) by utilizing physical invariance reasoning worked out in [2]. Indeed, according to the meaning of the rheological equation, it characterizes the processes occurring in a fixed material point moving as part of a continuum, and it should contain, in addition to material tensors and scalars, only observed dynamic and kinematic quantities connected (rigid) with just the given point of the continuum. In particular, the rheological equation should not depend on the parameters describing the rotation of the given material element during deformation, as well as the rotation of the whole continuum as a solid (rigid) unit. This is associated with the fact that the rotation of some particles of the medium depends essentially on the state of the adjacent particles of the medium. Applying this reasoning, we at once obtain (3.5) with all the resulting corollaries.

The relationships (2.5), (3.3) are utilized below to construct a rheological model of a compressible Maxwellian viscoelastic fluid with geometric and physical nonlinearity. A Maxwellian fluid is customarily defined as a medium with additive deformations, which are connected with the stresses in the flow by utilization of additional postulates.

4. It is customarily assumed [2] that the stress tensor p_{ij} in the elastic element of a Maxwellian model is connected with the elastic strain tensor by a linear relationship

$$p_{ij} = 2\mu\varepsilon_{(ij)} \tag{4.1}$$

Here μ is the shear modulus of the fluid. The relative change in volume associated with the elastic strain, can be written as

$$\varkappa = [(1-2\varepsilon_1)\ (1-2\varepsilon_2)\ (1-2\varepsilon_3)]^{-1/2} - 1 \tag{4.2}$$

Here ε_i $(i = 1, 2, 3)$ are the eigenvalues of the elastic strain tensor in the Lagrange basis.

In principle, as high a stress as desired can be realized in a viscoelastic medium. In particular, it can be considered that some eigenvalues of the tensor p_{ij} are greater than $\frac{1}{2}\mu$ so that the corresponding quantities $2\varepsilon_i$ are greater than unit. Recalling the definition of $2\varepsilon_{(ij)}$ we see that the strain described by Hooke's law (4.1) leads formally to a change in the signature of the metric tensor with all the resultant consequences. It is seen from (4.2), say, that in this case $\varkappa$ becomes complex. Analogous paradoxes results also from an attempt to consider energy processes in viscoelastic fluid flows.

Moreover, it is known that even for a generalized elastic shear strain the condition of conservation of the specific volume [3] should be satisfied. It follows from (4.1) and (4.2) that this natural requirement is not satisfied for finite deformations. For example, for simple shear we have

$$\varkappa = [1-(p_1/\mu)^2]^{-1/2} - 1 \neq 0$$

Here p_1 and $-p_1$ are eigenvalues of the stress tensor.

Even for small elastic deformations the assumption that the given viscoelastic medium possesses a multilateral compression modulus exactly equal to $2/3\mu$ actually corresponds to (4.1). Taking into account that real media possess a shear modulus on the order of 10^6–10^7 dyne/cm^2, we see that this assumption is equivalent to an assumption on very high compressibility of the medium. The compressibility of viscoelastic media in experiments is quite negligible, even at the pressure $p \gg \mu$.

The latter difficulty of a Maxwellian model is closely connected with the problem of describing an incompressible viscoelastic fluid. It can be considered 'incompressible', i.e., as not resulting in a change in specific volume, the total strains in the flow, or this requirement can be referred just to the viscous member of the total strains.

In the first case it must be implicitly assumed that, in general, the relative change in volume during elastic deformation is cancelled by a change in specific volume of opposite sign, which is connected with the viscous strain and occurs because of the effect of some additional physical factor.

In the second case it is assumed that the single reason for a possible change in specific volume, i.e., in compressibility of the medium, will be the elastic strains. The first hypothesis requires the introduction of some new compressibility mechanism, different from elastic strain (for example, the explicit consideration of structural changes in the moving fluid). The second hypothesis is perfectly natural within the scope of the expounded scheme, has sufficiently strong physical foundations, and can be formulated by analogy with the same hypothesis in the hydrodynamics of an incompressible viscous fluid.

Therefore, the necessity arises to replace the linear relationship (4.1) by some more adequate relationship without resulting in physical paradoxes. Let us note that, in principle, even Newton's law postulated for a viscous element, may be replaced by some nonlinear relationship between the stress tensors and the viscous strain rates. However, it can be assumed that such a substitution does not substantially affect the quality of the considered viscoelastic media, and leads only to a quantitative change in the characteristics of various flows.

5. Let us represent the stress tensor p_{ij} as

$$p_{ij} = -pg_{ij} + \tau_{ij}, \qquad P_1 = -3p + T_1 \tag{5.1}$$

Here p_1 and T_1 are the first invariants of the tensors p_{ij} and τ_{ij}, g_{ij} are metric tensor components. The first member on the right side of (5.1) describes purely reversible 'ideal' transfer of momentum in the system, and the second member defines the irreversible 'viscous' transfer of momentum. The most general linear relationship between τ_{ij} and the viscous strain rate tensor $\gamma_{[ij]}$ are

$$\tau_{ij} = 2\eta(\gamma_{[ij]} - {}^1/_3\Gamma_1 g_{ij}) + \zeta\Gamma_1 g_{ij}, \qquad \Gamma_1 = g^{ij}\gamma_{[ij]}$$

Here η and ζ are the fluid viscosity coefficients. As will be seen below $\Gamma_1 \equiv 0$. Hence it follows $T_1 = 3\zeta\Gamma_1 \equiv 0$. Taking the latter into account, we obtain Newton's law to connect the tensors $\gamma_{[ij]}$ and τ_{ij} in the form

$$\tau_{ij} = 2\eta\gamma_{[ij]} \tag{5.2}$$

The natural extension of (4.1) to a body with arbitrary continuity is represented in the physically linear theory by the relationship

$$\varepsilon_{ij} = \frac{P_1}{9K}g_{ij} + \frac{1}{2\mu}\left(p_{ij} - \frac{P_1}{3}g_{ij}\right), \qquad K = \lambda + \frac{2}{3}\mu \tag{5.3}$$

The bulk modulus K is defined in terms of the Lame coefficient λ and μ in such a way that the relative change in volume connected to the second member on the right side of (5.3) would equal zero identically for small deformations. It is possible to attempt a generalization of (5.3) to elastic incompressible media with finite deformations by replacing the first invariant p_1 of the tensor p_{ij} in (5.3) by a function of all three invariants P_i of this tensor such that the second member in the modified dependence (5.3) would not, as before, result in a change in the specific volume. This function has the meaning of a Lagrange multiplier in some variational problem [1]. Replacing $1/3\,P_1$ in (5.3) by $f(P_i)$ and allowing $K \to \infty$, we obtain the incompressibility condition as

$$(1-2\varepsilon_1)(1-2\varepsilon_2)(1-2\varepsilon_3) = 1$$

Introducing the nondimensional parameters $f = \mu\varphi$, $p_{ij} = \mu\pi_{ij}$ and the invariants

$$\Pi_1 = \pi_1 + \pi_2 + \pi_3, \quad \Pi_2 = \pi_1\pi_2 + \pi_2\pi_3 + \pi_3\pi_1, \quad \Pi_3 = \pi_1\pi_2\pi_3$$

we obtain an equation for φ

$$\varphi^3 + (3 - \Pi_1)\varphi^2 + (3 - 2\Pi_1 + \Pi_2)\varphi - (\Pi_1 - \Pi_2 + \Pi_3) = 0 \tag{5.4}$$

It is clear that it is necessary to take the root φ of this equation which equals $1/3\Pi_1$ for $\pi_i \ll 1$. It is easy to see that the strain described by the modified relationship (5.3) results, as before, in a change in the signature of the metric tensor. For example, let $\pi_1 = x$, $\pi_2 = \pi_3 = -{}^1/_2 x$, so that $\Pi_1 \equiv 0$. Then in the particular case $x = -3$, (5.4) has the solution $\varphi = 0$, and $1-2\varepsilon_2 < 0$, $1-2\,\varepsilon_3 < 0$, which also denotes a change in signature. Moreover, if the finite modulus K is again introduced, the pure shear strain results, as before, in a change in the specific volume. Hence, the considered phenomenological method of extending (5.3) to a system with finite deformations does not result in obtaining a consistent relation between the stress and elastic strain tensors.

Let the free energy of unit volume of a solid in the deformed state be $F(\varepsilon_{i}^{\cdot j})$. Considering a change dF in the quantity F for a change $d\varepsilon_{i}^{\cdot j}$ in the elastic strain tensor, let us write

$$dF = p^{i}_{\cdot j} d\varepsilon^{j}_{\cdot i} + {}^1/_2\,\lambda\,(\varepsilon_{i}^{\cdot j})\,(d\varepsilon^{i}_{\cdot i})^2 + \mu\,(\varepsilon_{i}^{\cdot j})\,d\varepsilon^{i}_{\cdot j} d\varepsilon^{j}_{\cdot i} \tag{5.5}$$

To simplify the writing, the parentheses in the subscript notation of the elastic strain tensor have been omitted here and below.

This expression is simply a series expansion of the increment in free elastic strain energy in the neighborhood of some deformed state characterized by the quantities $\varepsilon_{i}^{\cdot j}$. The possibility of selecting such a state as initial has been stressed in [1]. Evidently if $\varepsilon_{i}^{\cdot j} = 0$, i.e., a state in which there are no strains or residual stresses, has been selected as initial state, then the first term drops out in the right side of (5.5).

In principle, the most different forms of the function F can be utilized. We assume here that the coefficients λ and μ in (5.5) are constants. This assumption is equivalent to postulating 'homogeneity' of the relative strains, their independence of the strain in the undeformed state of the material. Applying standard methods to (5.5), we obtain

$$d\varepsilon_{i}^{\cdot j} = -\frac{dp}{3K}\delta_i^j + \frac{1}{2\mu}\left(dp_{i}^{\cdot j} - dp\,\delta_i^j\right) \tag{5.6}$$

In contrast to (5.3), which expresses the linearity of the strain process as a whole, the relationship (5.6) expresses the linearity of this process in the small. If this relationship were to admit of direct integration, the linear law (5.3) would then be obtained, or in the particular case $K = 2/3\mu$, the law (4.1). In the general case of finite deformations, such integration is impossible, because of the nonadditivity of the increments $d\varepsilon_{i}^{\cdot j}$ in successive deformations [1].

Let us formally represent the elastic strain process as a process passing through such intermediate states that the tensors ε_{ij}' in these states have the same principal axes as the tensor ε_{ij} in the final deformed state. From (5.6) we have the following Eq. with reference to the principal axes:

$$d\varepsilon_i = \frac{dl_i'}{l_i'} = -\frac{dp'}{3K} + \frac{1}{2\mu}(dp_i' - dp')$$

Here l_i' is the length of some linear element along the i-th principal axis in the intermediate deformed state, which equals l_i° in the initial and l_i in the final state. Integrating this equation we obtain

$$l_i = l_i^\circ \exp\left[-\frac{p}{3K} + \frac{1}{2\mu}(p_i - p)\right]$$

Using the definition of the relative elongations of material elements along the principal axes in terms of eigenvalues of the elastic strain tensor in the final state, and transforming from the equation for ε_i to the corresponding tensor equation, we obtain the following equation connecting ε_{ij} and p_j in the convective basis associated with the final deformed state:

$$\mathbf{E} = {}^1/_2(\mathbf{G} - e^{-2\mathbf{H}}), \quad \mathbf{E} = \|\varepsilon_{ij}\|, \quad \mathbf{P} = \|p_{ij}\|, \quad \mathbf{T} = \|\tau_{ij}\|, \quad \mathbf{G} = \|g_{ij}\| \qquad (5.7)$$

$$\mathbf{H} = -\frac{p}{3K}\mathbf{G} + \frac{1}{2\mu}(\mathbf{P} - p\mathbf{G}) = -\frac{p}{3K}\mathbf{G} + \frac{1}{2\mu}\mathbf{T}$$

Completely analogously, we obtain for the elastic strain tensor $\mathbf{E}_0$ defined in the space of initial states

$$\mathbf{E}_0 = {}^1/_2(e^{2\mathbf{H}_0} - \mathbf{G}_0) \qquad (5.8)$$

The inverse relationships to (5.7) are (5.9)

$$\mathbf{H} = -{}^1/_2 \ln(\mathbf{G} - 2\mathbf{E}), \quad \mathbf{P} = -\mu \ln(\mathbf{G} - 2\mathbf{E}) + 3\alpha p\mathbf{G}, \quad \alpha = {}^1/_3({}^2/_3\,\mu/k - 1)$$

The 'true' strain tensor $\mathbf{H}$ has repeatedly been utilized earlier in the theory of finite elastic deformations, in particular, Hencky postulated a linear connection between $\mathbf{P}$ and $\mathbf{H}$. From (5.9), we have for the first invariants $\mathbf{P}_1$ and $\mathbf{E}_1$ of the tensors p_{ij} and ε_{ij}

$$P = -3p = -\mu(1 + 3\alpha)^{-1} \ln[(1 - 2\varepsilon_1)(1 - 2\varepsilon_2)(1 - 2\varepsilon_3)]$$

$$E_1 = {}^1/_3(3 - e^{-2h_1} - e^{-2h_2} - e^{-2h_3}), \qquad h_i = (2\mu)^{-1}(p_i - 3\alpha p) \qquad (5.10)$$

The compressibility of a viscoelastic medium characterized by the relationships (5.7) to (5.10) is determined entirely by the magnitude of the modulus K when the hypothesis used in Section 4 that the sole reason for the change in the specific volume of a medium is its elastic strain, is satisfied. It is sufficient to consider just the hydrostatic compression of such a material. For $p_i = -p$ ($i = 1, 2, 3$) we have

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \frac{1}{2}\left(1 - \exp\frac{2p}{3K}\right), \qquad \varkappa = \exp\frac{-p}{K} - 1$$

On the other hand, the shear strains do not generally cause a change in volume for any K, as should be. For example, for simple shear we obtain

$$\varepsilon_1 = \frac{1}{2}\left(1 - \exp\frac{-p_{12}}{\mu}\right), \qquad \varepsilon_2 = \frac{1}{2}\left(1 - \exp\frac{p_{12}}{\mu}\right), \qquad \varepsilon_3 = 0, \qquad \varkappa = 0$$

Evidently the eigenvalues of the tensor $\mathbf{H}$ are always negative, so that the paradoxes associated with the change in signature of the metric tensor do not generally arise in this case.

6. Utilizing the results of Section 3, let us represent the kinematic relationship (2.5) as

$$(\partial/\partial t + v^m\nabla_m)e_{ij} + e_{im}\nabla_j v^m + (\nabla_i v^m)e_{mj} +$$
$$+ (\delta_i^m - 2e_{i\cdot}^{\cdot m})\gamma_{[mj]} = \gamma_{ij} = {}^1/_2(\nabla_i v_j + \nabla_j v_i) \qquad (6.1)$$

Electring the elastic strain tensor components and the velocities of total displacement of the medium as governing variables, we obtain a rheological equation from (6.1) when (5.2) and (5.9) are taken into account

$$2\eta\,[(\partial/\partial t + v^m\nabla_m)\,e_{ij} + e_{im}\nabla_j v^m + (\nabla_i v^m)\,e_{mj}] -$$
$$-\mu\,(\delta_i^m - 2e_{i\cdot}^{\cdot m})\,\{\ln(\mathrm{G} - 2\mathrm{E})\}_{mj} + (1+3\alpha)\,p\,(g_{ij} - 2e_{ij}) = 2\eta\gamma_{ij} \quad (6.2)$$

Taking account of the definition of E in terms of **P** or **T** according to (5.7), it is also easy to obtain a rheological equation in terms of the variables p_{ij} or τ_{ij} from (6.2).

The components of the Hencky tensor **H** can be expressed in terms of e_{ij} by utilizing the Lagrange-Sylvester formulas. Moreover, using the Hamilton-Cayley theorem, we arrive at a quadratic equation in e_{ij} (or in p_{ij}, τ_{ij}) with coefficients dependent on invariants of the tensor **E**.

For small elastic strains the tensor **H** may be expanded in a series in **E**. We then obtain an approximate from (6.2)

$$\eta\,[(\partial/\partial t + v^m\nabla_m)\,e_{ij} + e_{im}\nabla_j v^m + (\nabla_i v^m)\,e_{mj}] +$$
$$+(\delta_i^m - 2e_{i\cdot}^{\cdot m})\,[\mu e_{mj} + {}^1\!/_2\,(1+3\alpha)\,p g_{mj}] = \eta\gamma_{ij} \quad (6.3)$$

Or in terms of τ_{ij}, this Eq. is (we use the approximate relationship $\mathrm{H} \approx \mathrm{E}$) following from (5.7))

$$\theta\,[(\partial/\partial t + v^m\nabla_m)\,p_{ij}' + p_{im}'\nabla_j v^m + (\nabla_i v^m)\,p_{mj}'] +$$
$$+(\delta_i^m - \mu^{-1}p_{i\cdot}'^{\cdot m})\,\tau_{mj} = 2\eta\gamma_{ij}, \qquad p_{ij}' = \tau_{ij} - (1+3\alpha)\,p g_{ij} \quad (6.4)$$

Here θ is the single relaxation time of the fluid.

For an incompressible fluid $K \to \infty$ and $\alpha = -1/3$. Hence, (6.2) becomes

$$2\eta\,[(\partial/\partial t + v^m\nabla_m)\,e_{ij} + e_{im}\nabla_j v^m + (\nabla_i v^m)\,e_{mj}] -$$
$$-\mu\,(\delta_i^m - 2e_{i\cdot}^{\cdot m})\,\{\ln(\mathrm{G} - 2\mathrm{E})\}_{mj} = 2\eta\gamma_{ij}$$

For an incompressible fluid (6.4) is written in the form

$$\theta\,[(\partial/\partial t + v^m\nabla_m)\,\tau_{ij} + \tau_{im}\nabla_j v^m + (\nabla_i v^m)\,\tau_{mj}] + (\delta_i^m - \mu^{-1}\tau_{i\cdot}^{\cdot m})\,\tau_{mj} = 2\eta\gamma_{ij}$$

It can be expected that the compressibility of a viscoelastic medium will turn out to be particularly essential for the investigation of the stability of various stationary flows because it may, in principle, cause the appearance of elastic compression-rarefaction waves, different from acoustic waves, in the flow, which may in turns stimulate the development of the customary hydrodynamic instability. A possible role of the effects of compressibility in the development of hydrodynamic instability in viscoelastic fluid flow is discussed in [4].

Let us note that the quantity p introduced in (5.1) and playing the part of the effective external pressure is in the rheological equation. This is connected with the fact that for a nonlinear dependence between the elastic strains and the stresses, the capacity of the fluid to further absolute elastic strain depends on the already existing strains. This latter permits the comprehension, to some degree, of known experiments on stabilization of viscoelastic fluid flow with the rise in external pressure.

Let us turn to the kinematic relationship (6.1) with the tensor g^{ij}. Taking account of (3.3), we hence obtain the equality

$$g^{ij}\gamma_{(ij)} + g^{ij}\gamma_{[ij]} = g^{ij}\gamma_{ij} \quad (6.5)$$

The right side here is the rate of change of the specific volume of fluid, the first and second terms in the left side of (6.5) are the rates of change in specific volume due to the elastic and viscous strains, respectively. According to the assumptions made in Section 4, $\Gamma_1 = g^{ij}\gamma_{[ij]} \equiv 0$, which justifies the formulation of Newton's law in the form (5.2). The first member on the left side of (6.5) satisfies an equation which results from (3.3)

$$g^{ij}\gamma_{(ij)} = (\partial / \partial t + v^m \nabla_m)\, g^{ij} e_{ij} + 2 g^{ij} e_{im} \gamma^{(m\cdot)}_{(\cdot j)}$$

It is hence seen that $g^{ij}\gamma_{(ij)} = g^{ij}\gamma_{ij}'$, where γ_{ij}' satisfy the matrix Eq.

$$(\mathrm{G} - 2\mathrm{E})\,\gamma' = (\partial / \partial t + v^m \nabla_m)\,\mathrm{E}, \qquad \gamma' = \| \gamma_{ij}' \|$$
$$\gamma' = (\mathrm{G} - 2\mathrm{E})^{-1} (\partial / \partial t + v^m \nabla_m)\,\mathrm{E} \tag{6.6}$$

Here it has explicitly been taken into account that the metric tensor of the Euler coordinate system is independent of time.

For the fluid density ρ, we therefore have Eq.

$$\frac{1}{\rho}\left(\frac{\partial}{\partial t} + v^m \frac{\partial}{\partial x^m}\right)\rho = - g^{ij}\gamma_{(ij)} = - \{(\mathrm{G} - 2\mathrm{E})^{-1}\}^{ij} \left(\frac{\partial}{\partial t} + v^m \nabla_m\right) e_{ij} \tag{6.7}$$

Eq. (6.7) plays the part of the thermodynamic equation of state of a viscoelastic medium. Let us note that by virtue of (6.5), Eq. $g^{ij}\gamma_{[ij]} = 0$ follows directly from (6.7) and the continuity equation for the total deformations of a viscoelastic fluid.

7. Besides the rheological equation, we have the dynamic Navier-Stokes equations and the continuity equation for the description of viscoelastic fluid motion. In the Euler coordinate system they are:

$$\rho\left(\frac{\partial}{\partial t} + v^m \nabla_m\right) v_i = - \nabla^m p_{im} + \rho f_i, \qquad \frac{\partial \rho}{\partial t} + \nabla^m (\rho v_m) = 0 \tag{7.1}$$

It is easy to see that (7.1) together with (6.2) and (6.7) form a complete system of equations to determine the ten unknowns v_i, τ_{ij}, p and ρ which describe the mechanical behavior of the medium. There are in all five independent quantities τ_{ij} since they are connected by the condition $T_1 = 0$ which follows from Eq. $g^{ij}\gamma_{[ij]} = 0$.

From the first equations of (7.1), we obtain by the customary means [1], an energy equation in which all the quantities are referred to unit volume of medium

$$dE = dA_1 + dA_2 + dA_3, \quad dE = \rho v^i dv_i \quad f_i = f_i^{(1)} + f_i^{(2)}$$
$$dA_1 = \rho f_i^{(1)} v^i dt + dA', \quad dA_2 = \rho f_i^{(2)} v^i dt, \quad dA_3 = - p_{ij}\gamma^{ij} dt \tag{7.2}$$

Here dE is the change in kinematic energy of the fluid, dA_1 is the work of all the external forces, where dA' is the work of the external surface forces, dA_2 the work of the potential volume forces, and dA_3 the elementary work of the internal surface forces. The quantities $f_i^{(1)}$ and $f_i^{(2)}$ are the nonpotential and potential mass forces, respectively.

Taking account of (3.3) we obtain from (6.1)

$$- dA_3 = p^{ij}\gamma_{ij}\, dt = p^{ij}\gamma_{(ij)}\, dt + p^{ij}\gamma_{[ij]}\, dt \tag{7.3}$$

The first member on the right side of (7.3) is the elementary work of the internal stresses on the elastic strains of the medium and equals the increment dF in the free energy of elasticity. The second member on the right in (7.3) describes viscous energy dissipation Wdt. Using (5.2), we obtain an expression

$$W = (2\eta)^{-1} p^{ij}\tau_{ij} = (2\eta)^{-1} \tau^{ij}\tau_{ij} \tag{7.4}$$

The identity $g^{ij}\tau_{ij} \sim g^{ij}\gamma_{[ij]} \equiv 0$ was also taken into account here.

For the increment in free energy of elasticity, we obtain from (3.3)

$$dF = p^{ij}\gamma_{(ij)}\, dt = p^{ij} (\partial / \partial t + v^m \nabla_m)\, e_{ij} dt + 2 p^{ij} e_{im} \gamma^{(m\cdot)}_{(\cdot j)}\, dt$$

As in Section 3, it is easy to see that $p^{ij}\gamma_{(ij)} = p^{ij}\gamma_{(ij)}'$, where the matrix of the coefficients of γ_{ij}' is determined by (6.6). Hence, we have

$$dF = p^{ij} \{(\mathrm{G} - 2\mathrm{E})^{-1}\}_{j\cdot}^{\cdot m} (\partial / \partial t + v^l \nabla_l)\, e_{mi}\, dt$$

Inserting the components h_{ij} of the tensor **H** from (5.9), we hence obtain

$$dF = 2\mu h^{ij}\, dh_{ij} + 3\alpha p g^{ij}\, dh_{ij} \tag{7.5}$$

In particular, from (7.5) follows the free energy Eq.

$$F = \text{const} + \mu\,(h_1^2 + h_2^2 + h_3^2) + 3\alpha p\,(h_1 + h_2 + h_3) \tag{7.6}$$

Here h_i are eigenvalues of the tensor **H** defined in (5.10). For an incompressible fluid the last members in (7.5) and (7.6) vanish since the sum of the quantities h_i equals zero for $\alpha = -1/3$.

It is analogously easy to determine the other thermodynamic functions associated with the elastic strain.

Substituting (7.3) into the energy equation (7.2), we obtain a new Eq.

$$dA_1 + dA_2 = dE + dF + Wdt \tag{7.7}$$

This Eq. expresses the fact that the elementary work of all the external forces equals the sum of the increments in the kinematic energy of the fluid, the free energy of the elastic strain, and the viscous energy dissipation. Let us write the first law of thermodynamics

$$dE + dU + d\Psi = dA_1 + dQ$$

Here dU is the change in internal energy of a viscoelastic medium $d\Psi$ is the change in its potential energy in the force field $f_i^{(2)}$, dQ is the heat flux; as before, all the quantities are referred to unit volume of the medium. Substituting dE from (7.2) herein, and taking into account that $dA_2 + d\Psi = 0$, we obtain the heat flux Eq. in the form

$$dQ = dU + dA_3$$

Using the expression $dU = TdS + dF$, and the representation of dA_3 from (7.3), we finally obtain

$$TdS = dQ + Wdt \tag{7.8}$$

Let us note that if the elasticity of the considered medium is purely entropic in nature, as is intrinsic to the majority of rubberlike materials, then we can consider $dU \approx 0$ in all the above equations. We then have

$$dF \approx TdS = dQ + Wdt \tag{7.9}$$

The presented energy relationships refer primarily to isothermal flows of a viscoelastic Maxwellian fluid. For other flows (adiabatic, say), corresponding relationships may be obtained by standard means [1].

Let us note the fundamental qualitative singularities of Maxwellian fluid flows considered herein. As can be shown, by considering concrete flows of this fluid, the obtained rheological equation describes both the appearance of normal stresses in different flows, and also that the flow curve is non-Newtonian (the dependence of the effective viscosity on the shear velocity). It is hence essential that the first invariant of the tensor r_{ij} be always zero for the considered model.

This means that if an additional tensile stress acts in some direction on the moving fluid, then it equals the absolute value of the compressive normal stress acting in a perpendicular direction. For example, in plane stationary Couette flow the fluid is stretched along the stream, and compressed in a direction perpendicular to the plates.

It is clear that these singularities of the considered medium are mainly connection with the assumptions made in Section 5 informulating the tensor relations (5.7) and (5.8), which result in a quadratic dependence of the free elasticity energy (7.5) on the components of the Hencky tensor. A more complex expression can certainly be given for F, dependent on higher degrees of h_i, say. Then the first invariant of the tensor r_{ij} would turn out to be nonzero in the general case. The choice of the function F corresponding to some real class of viscoelastic media, as well as the extension of the simplest model to media with discrete or continuous relaxation time spectrum is an independent problem.

Let us just note that the rheological equations of Section 6 are very similar to that obtained by De Witt [5] on the basis of a formal generalization of the linear Maxwell equation by using the Jaumann derivative. There is nothing surprising here, since De Witt postulated expressions for the elastic strain rate tensor components which differ from (3.3) just by the absence of terms with $\gamma_{(ij)}$ in the right side. It could hence be expected that both models should result in identical or very similar results in a number of cases.

In an analogous problem Oldroyd [2] also started from the relationshi (2.2), but he identified the convective derivatives of the viscous and elastic strain tensors with the tensors of the corresponding strain rates.

Considering the mixed or contravariant tensor components as unknowns, Oldroyd obtained rheological equations describing a fluid with essential different properties [2]. This ambiguity would not have arisen in [2], had he used the correct relationships following from [1] in place of the explicitly incorrect relationships of the type (2.1), written for components with contravariant or mixed configuration of the indices. For example, for the mixed components the additivity relationship is written as

$$\varepsilon_{i\cdot}^{\cdot j} = \varepsilon_{(i\cdot)}^{(\cdot j)} + \varepsilon_{[i\cdot]}^{[\cdot j]} - 2\varepsilon_{(i\cdot)}^{(\cdot m)}\varepsilon_{[m\cdot]}^{[\cdot j]}$$

As a result of a computation based on this relationship, we arrive at equations which differ from those obtained above by just multiplication by a contravariant metric tensor. Moreover, Hooke's law in the form (4.1), which is incorrect for large elastic deformations, is used in the Oldroys theory.

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Translated by M.D.F.